Electrodynamics of Rotating Bodies

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Contents

Abstract

In this paper, we aim to investigate the electrodynamics of rotating bodies. First, we try to derive Maxwell's equations in a rotating frame of reference. Then, to further explore a rotating system, we consider an ideal rotating dipole. In many reference books, this problem is solved using the method of two orthogonal dipoles, one stationary and the other changing. Here, we calculate the electric potential and vector potential in the general form using the Liénard-Wiechert potentials through the relativistic method. We then derive the general electric potential for a rotating ideal dipole by writing these equations. Next, we show that in the special case, this leads to the well-known results found in reference books.

For further investigation, we examine the radiation of this dipole and calculate its dissipative power. The exciting part of the story begins here! Using the general formula we derived for the electric potential of an ideal rotating dipole, we calculate the equipotential surfaces of this rotating ideal dipole. We observe that the graphs generated using this method surprisingly match the results of a simulation based on another approach. We have also verified this agreement using a different method! Finally, we conclude with a historical analogy and discuss alternative methods and other ideas for expanding this paper.

1 Maxwell's Equations in the Rotating Frame

First, we attempt to derive Maxwell's equations in a rotating frame. In view of future applications, in these calculations we assume that the speed is much smaller than the speed of light, so that terms of order $\frac{\nu^2}{c^2}$ and higher can be neglected. This is because later we will examine an ideal dipole rotating with a limited angular speed. In this example, due to the ideal nature (with the rotation distance tending to zero and the angular speed being limited), the speed of each particle tends to zero relative to the speed of light—in other words, the wavelength associated with the emitted wave is much larger than the dimensions of the dipole. This approximation is very useful; one advantage is that we can approximate the relativistic effects. Also, given the ideal nature of the dipole, its dimensions vanish and the centripetal acceleration becomes negligible, which greatly aids in idealizing and "smoothing" the system.

According to this approximation, we take the Lorentz factor to be 1 in all these calculations, because the term under the square root is of second order with respect to the discussed approximation and is thus neglected. Since there is no translational motion between the two frames and given the above approximation, we can neglect the Lorentz force (although, in general, this force does transform under relativistic transformations; but due to the approximation—taking the gamma factor as 1 and neglecting higher order terms in the ratio of speed to the speed of light—we can make this assumption). Moreover, the smallness of the speed compared to the speed of light allows us to ignore time dilation, length contraction, and issues such as the Ehrenfest paradox.

Now, let us assume that we have a rotating frame which rotates relative to a stationary frame (with a common origin) with an angular speed ω about the z-axis, i.e. the motion is purely rotational with no translation. In this section we will use primed coordinates to refer to the rotating frame. For simplicity in calculations and notation, we assume that both the vacuum permittivity and the vacuum permeability are equal to 1, so that the speed of light is also 1. Moreover, since the Gaussian system is common, especially in relativity, we also adopt it here.

Since later we assume that the dipole rotates in a plane perpendicular to the z-axis, our chosen frame is effectively a two-dimensional rotating frame with angular speed ω about the z-axis; this leads us to use flat polar coordinates (in effect, the angle θ is 90 degrees—indeed, we use cylindrical coordinates).

Given the approximations and issues mentioned, the transformation between the two coordinate systems is as follows:

$$
\begin{aligned}
r &= r' \\
\phi' &= \phi - \omega t \\
z' &= z \\
t' &= t
\end{aligned}
\tag{1}
$$

Clearly, this transformation preserves the volume; hence, by virtue of charge conservation, the transformations for current density and charge density are:

$$
\begin{aligned}
\rho' &= \rho \\
\vec{J'} &= \vec{J} - \rho\vec{\nu}
\end{aligned}
\tag{2}
$$

where $\vec{\nu}$ is the velocity of the observer relative to the laboratory frame. Based on the above approximations and using the relativistic transformation of the fields, we have:

$$\vec{E'} \approx \vec{E} + \frac{\vec{\nu}}{c} \times \vec{B} \tag{3}$$

$$\vec{B'} \approx \vec{B} \tag{4}$$

Now, we try to formulate Maxwell's equations in the rotating frame. Since, according to equation 1, the intervals of length, area, and volume are the same in both frames, we have:

$$\vec{\nabla}' = \vec{\nabla} \tag{5}$$

so that clearly the equation $\vec{\nabla} \cdot \vec{B} = 0$ becomes:

$$\vec{\nabla}' \cdot \vec{B}' = 0 \tag{6}$$

Using equation 4, it is straightforward to obtain from $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ and the above transformations the equation:

$$\vec{\nabla}' \cdot \vec{E}' = 4\pi\rho' + \vec{\nabla}' \cdot \left(\frac{\vec{\nu}}{c} \times \vec{B}'\right) = 4\pi\rho' + \vec{B}' \cdot \left(\vec{\nabla}' \times \frac{\vec{\nu}}{c}\right) - \frac{\vec{\nu}}{c} \cdot \left(\vec{\nabla}' \times \vec{B}'\right) \tag{7}$$

Considering equations 2, 3, 4 and noting that $\vec{\nu} = \vec{\omega} \times \vec{x'}$, one obtains:

$$\vec{\nabla}' \times \vec{\nu} = \vec{\omega}(\vec{\nabla}' \cdot \vec{x'}) - (\vec{\omega} \cdot \vec{\nabla}')\vec{x'} = 2\vec{\omega}$$

Thus, equation 7 becomes:

$$\vec{\nabla}' \cdot \vec{E}' = 4\pi\rho' + \left(\frac{2\vec{\omega} \cdot \vec{B}'}{c}\right) - \frac{\vec{\nu}}{c} \cdot \left(\vec{\nabla}' \times \vec{B}'\right) \tag{8}$$

It is clear that by taking into account the relativistic time transformation we could obtain further details; however, given the approximations mentioned, we neglect these higher-order effects. Now, we attempt to find a relation between the time derivatives of the vector potential in the two frames:

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{\nu} \cdot \vec{\nabla})\vec{A} \tag{9}$$

$$\frac{d\vec{A}}{dt'} = \frac{\partial \vec{A}}{\partial t'} + (\vec{\nu'} \cdot \vec{\nabla}')\vec{A}$$

Since in the attached frame $\vec{\nu'} = 0$, it follows that:

$$\frac{d\vec{A}}{dt'} = \frac{\partial \vec{A}}{\partial t'}$$

Using equation 9 and noting that, since the frames only rotate relative to one another and the radial component of the velocity is zero (so that its divergence is zero, which can be verified directly), we have:

$$\frac{\partial \vec{A}}{\partial t} = \frac{\partial \vec{A}}{\partial t'} + \vec{\omega} \times \vec{A} - (\vec{\nu} \cdot \vec{\nabla})\vec{A} \tag{10}$$

In Cartesian coordinates (and not in other coordinate systems such as spherical or cylindrical), we can use the following relations:

$$\vec{\omega} \times \vec{A} = \vec{\omega} \times \left((\vec{A} \cdot \vec{\nabla})\vec{x}\right) = (\vec{A} \cdot \vec{\nabla})(\vec{\omega} \times \vec{x}) = (\vec{A} \cdot \vec{\nabla})\vec{\nu}$$

$$\vec{\nabla} \times (\vec{\nu} \times \vec{A}) = \vec{\nu}(\vec{\nabla} \cdot \vec{A}) - \vec{A}(\vec{\nabla} \cdot \vec{\nu}) + (\vec{A} \cdot \vec{\nabla})\vec{\nu} - (\vec{\nu} \cdot \vec{\nabla})\vec{A} = \vec{\nu}(\vec{\nabla} \cdot \vec{A}) + \vec{\omega} \times \vec{A} - (\vec{\nu} \cdot \vec{\nabla})\vec{A}$$

Thus, equation 10 takes the form:

$$\frac{\partial \vec{A}}{\partial t} + \vec{\nu}(\vec{\nabla} \cdot \vec{A}) = \frac{\partial \vec{A}}{\partial t'} + \vec{\nabla} \times (\vec{\nu} \times \vec{A}) \tag{11}$$

We can write the partial derivative of the magnetic field as:

$$\frac{\partial \vec{B}}{\partial t} = \frac{\partial \vec{B}}{\partial t'} + \vec{\nabla} \times (\vec{\nu} \times \vec{B}) = \frac{\partial \vec{B'}}{\partial t'} + \vec{\nabla}' \times (\vec{\nu} \times \vec{B'})$$

Thus, we have:

$$\vec{\nabla} \times \vec{E} = \vec{\nabla}' \times \left(\vec{E}' - \frac{\vec{\nu}}{c} \times \vec{B}' \right) = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} = -\frac{1}{c} \frac{\partial \vec{B}'}{\partial t'} - \vec{\nabla}' \times \left(\frac{\vec{\nu}}{c} \times \vec{B}' \right)$$

which implies:

$$\vec{\nabla}' \times \vec{E}' = -\frac{1}{c} \frac{\partial \vec{B}'}{\partial t'} \tag{12}$$

Using equation 11 and the divergence equation for the electric field in Maxwell's equations, as well as the transformation in 3, one can compute the time derivative of the electric field:

$$\frac{\partial \vec{E}}{\partial t} + 4\pi\rho\vec{\nu} = \frac{\partial \vec{E}}{\partial t'} + \vec{\nabla} \times (\vec{\nu} \times \vec{E}) = \frac{\partial \vec{E}'}{\partial t'} - \frac{\vec{\nu}}{c} \times \frac{\partial \vec{B}'}{\partial t'} + \vec{\nabla}' \times \left(\vec{\nu} \times \left(\vec{E}' - \frac{\vec{\nu}}{c} \times \vec{B}' \right) \right)$$

Finally, we have:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla}' \times \vec{B}' = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{J}' + \frac{1}{c} \frac{\partial \vec{E}'}{\partial t'} - \frac{\vec{\nu}}{c^2} \times \frac{\partial \vec{B}'}{\partial t'} + \vec{\nabla}' \times \left[\frac{\vec{\nu}}{c} \times \left(\vec{E}' - \frac{\vec{\nu}}{c} \times \vec{B}' \right) \right] \tag{13}$$

Thus, we have derived all of Maxwell's equations for the rotating frame (equations 6, 8, 12 and 13). In the following, we will examine a specific rotating system, namely, an ideal rotating dipole. In the next section, we first derive the Liénard-Wiechert potentials using a relativistic method.

2 Relativistic Calculation of the Liénard-Wiechert Potentials

In this section, we aim to calculate the Liénard-Wiechert potentials using a relativistic method. To do so, we go to the rest frame attached to the body; in that frame, since the particle is at rest and the situation is static, the electric potential is obtained from the well-known electrostatic equation. Of course, one must note that if the frame were accelerating, one should use the instantaneous comoving frame. However, in the example we will consider later, due to the small size of the dipole and the approximations discussed, the retardation effects are negligible and unimportant. (In the notation of this article, an asterisk denotes quantities in the comoving frame of the particle, while quantities without an asterisk are in the laboratory frame; also, due to the importance of retardation time in the relativistic calculation, only in this section do we explicitly denote the retardation time with brackets, while in the remainder of the article it is assumed and not repeatedly written.) According to the relativistic transformations of the fields as known from special relativity, the electric potential and the vector potential form a four-vector that must transform under Lorentz transformations (again, note the aforementioned approximation for the instantaneous frame—even though one could neglect this approximation and choose the frame such that it is comoving with the body and instantaneous, as is done in rigid body mechanics for writing Euler's equations). Thus, we can write the following equations:

$$V = [\gamma]\Big(V^{\star} + [\vec{\beta}c] \cdot \vec{A}^{\star}\Big) = \frac{1}{4\pi\epsilon_0} q \Big[\frac{\gamma}{R^{\star}}\Big] \tag{14}$$

$$\vec{A}_{\parallel} = [\gamma]\Big(\vec{A}^{\star}_{\parallel} + \Big[\frac{\vec{\beta}}{c}\Big] V^{\star}\Big) = \frac{1}{4\pi\epsilon_0 c} q \Big[\frac{\gamma\vec{\beta}}{R^{\star}}\Big] \tag{15}$$

$$\vec{R} = \vec{r} - \vec{r'} \tag{16}$$

where $\vec{r}$ and $\vec{r'}$ are, respectively, the observation point of the potential and the position of the moving particle.

It is evident that the perpendicular components of the vector potential in both frames vanish.

$$\vec{A}_{\parallel} = (\vec{A} \cdot [\hat{\beta}])[\hat{\beta}] \tag{17}$$

$$\vec{A}_\perp = \vec{A} - \vec{A}_\parallel = \vec{A} - (\vec{A}\cdot[\hat{\beta}])[\hat{\beta}] \tag{18}$$

$$\left[R^\star\right] = \Delta x_0^\star = [\gamma]\Big(\Delta x_0 - [\vec{\beta}]\cdot\Delta\vec{x}\Big) = [\gamma(R - \vec{\beta}\cdot\vec{R})] = [\gamma(1 - \vec{\beta}\cdot\hat{R})R] \tag{19}$$

By substituting equation 19 into equation 14, we obtain the Liénard-Wiechert electric potential.
Similarly, by substituting equation 19 into equation 15 and using equation 18 along with the fact that the perpendicular component of the vector potential is zero, the Liénard-Wiechert vector potential is obtained:

$$V = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \frac{\vec{R}\cdot\vec{\nu}}{c}} \tag{20}$$

$$\vec{A} = \frac{\vec{\nu}}{c^2}V \tag{21}$$

All parameters on the right-hand side of each equation are evaluated at the retarded time.

3 Calculation of the Fields Due to an Ideal Rotating Dipole

Now, based on the Liénard-Wiechert potentials, we attempt to calculate the resultant potential of an ideal dipole. Assume that the dipole rotates counterclockwise, and further assume that the plane of rotation is the x-y plane (i.e. ω is directed along the z-axis).

$$V_\pm = \frac{\pm q}{4\pi\epsilon_0}\frac{1}{R_\pm - \frac{\vec{R}_\pm\cdot\vec{\nu}_\pm}{c}} \tag{22}$$

where all parameters are evaluated at the retarded time defined as:

$$t_r = t - \frac{R}{c} \tag{23}$$

$$\vec{r'}_{\pm} = \pm\frac{a}{2}\Big(\cos(\omega t_r)\,\hat{x} + \sin(\omega t_r)\,\hat{y}\Big) \tag{24}$$

$$\vec{\nu}_{\pm} = \vec{\omega} \times \vec{r'}_{\pm} \tag{25}$$

Considering equations 24 and 25 and noting that the angular velocity is taken along the z-axis, the velocity of each dipole charge is:

$$\vec{\nu}_{\pm} = \pm\frac{\omega a}{2}\Big(-\sin(\omega t_r)\,\hat{x} + \cos(\omega t_r)\,\hat{y}\Big) \tag{26}$$

$$R_{\pm} = \sqrt{r^2 + r'^2_{\pm} - 2\vec{r}\cdot\vec{r'}_{\pm}} \tag{27}$$

Using equation 27 and equation 24, we can write:

$$R_{\pm} = \sqrt{r^2 + \frac{a^2}{4} \pm a\Big(x\,\cos(\omega t_r) + y\,\sin(\omega t_r)\Big)} \tag{28}$$

where x and y are the contravariant coordinates of the vector $\vec{r}$, i.e.:

$$\vec{r} = (x, y, z)$$

$$\frac{-\vec{R}_{\pm}\cdot\vec{\nu}_{\pm}}{c} = -\frac{(\vec{r} - \vec{r'}_{\pm})\cdot\vec{\nu}_{\pm} = \vec{r}\cdot\vec{\nu}_{\pm}}{c} = \frac{\mp\frac{\omega a}{2}\Big(-x\,\sin(\omega t_r) + y\,\cos(\omega t_r)\Big)}{c} \tag{29}$$

$$V_{\pm} = \frac{\pm q}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2 + \frac{a^2}{4} \pm a\Big(x\,\cos(\omega t_r) + y\,\sin(\omega t_r)\Big)} \pm \frac{\omega a}{2c}\Big(-x\,\sin(\omega t_r) + y\,\cos(\omega t_r)\Big)} \tag{30}$$

$$V_{\pm} \approx \frac{\pm q}{4\pi\epsilon_0}\frac{1}{r \mp \frac{a}{2r}\Big(x\,\cos(\omega t_r) + y\,\sin(\omega t_r)\Big) \mp \frac{\omega a}{2c}\Big(-x\,\sin(\omega t_r) + y\,\cos(\omega t_r)\Big)} \tag{31}$$

$$V_{\pm} \approx \frac{\pm q}{4\pi\epsilon_0 r}\left(1 \pm \frac{a}{2r^2}\Big(x\,\cos{(\omega t_r)} + y\,\sin{(\omega t_r)}\Big) \pm \frac{\omega a}{2cr}\Big(-x\,\sin{(\omega t_r)} + y\,\cos{(\omega t_r)}\Big)\right) \tag{32}$$

At this stage, we must employ the aforementioned approximations. For an ideal dipole, we assume that the dipole size is small compared to the emitted wavelength and also that the distance at which the potential is measured is small; in fact, for an ideal dipole this assumption is exact since all higher order terms vanish.

One may also assume that the wavelength associated with the emitted electromagnetic wave is small compared to the measurement distance. This assumption has been made in some references for certain formulas, though one can also compute the general potential of a rotating dipole without this assumption.

First, neglecting the third assumption and performing the calculations, we note that by the superposition principle the resultant potential is the sum of the potentials due to the positive and negative charges, i.e.,

$$V = V_+ + V_- \tag{33}$$

$$V \approx \frac{qa}{4\pi\epsilon_0 r^2}\left(\frac{1}{r}\Big(x\,\cos{(\omega t_r)} + y\,\sin{(\omega t_r)}\Big) + \frac{\omega}{c}\Big(-x\,\sin{(\omega t_r)} + y\,\cos{(\omega t_r)}\Big)\right) \tag{34}$$

This equation is the general expression for the scalar potential of an ideal rotating dipole. Recall that the dipole moment is defined as the product of the charge magnitude and the separation between the charges, and by converting from Cartesian to polar coordinates, x and y can be expressed in terms of the polar parameters. Now, if we use the third assumption in the approximation, i.e. assume that the emitted wavelength is much smaller than the measurement distance, we obtain:

$$V \approx -\frac{p_0\omega}{4\pi\epsilon_0 c}\Big(\frac{\sin\theta}{r}\Big)\Big(\cos\phi\,\sin{(\omega t_r)} - \sin\phi\,\cos{(\omega t_r)}\Big) \tag{35}$$

Taking into account the retardation time and the approximations mentioned, we have:

$$V \approx -\frac{p_0\omega}{4\pi\epsilon_0 c}\Big(\frac{\sin\theta}{r}\Big)\Big(\cos\phi\,\sin\left(\omega t-\omega\frac{r}{c}\right)-\sin\phi\,\cos\left(\omega t-\omega\frac{r}{c}\right)\Big) \tag{36}$$

This equation can be readily obtained by considering two orthogonal, time-varying stationary dipoles. Note that the dipole moment of a rotating dipole is given by

$$\vec{p} = p_0\Big(\cos\left(\omega t_r\right)\hat{x}+\sin\left(\omega t_r\right)\hat{y}\Big)$$

Intuitively, this equation means that there are two dipoles—one along the x-axis and another along the y-axis—with the charge magnitude on each varying with a specific phase (one may assume that these two vary with a phase difference of $\frac{\pi}{2}$, i.e. they oscillate out of phase). It is evident that if one uses the retarded potential for a stationary and time-varying dipole ($V \approx -\frac{p_0\omega}{4\pi\epsilon_0 c}\Big(\frac{1}{r}\Big)(z\,\sin\left(\omega t-\omega\frac{r}{c}\right))$) and, noting that instead of the z-axis the x and y axes must be used (since the stationary dipoles are along these axes) and also considering that these two dipoles have a phase difference of $\frac{\pi}{2}$, then by applying the superposition principle we again obtain equation 36.
Based on the Liénard-Wiechert equations, we can easily calculate the vector potential as well. Moreover, we know:

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}V \tag{37}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{38}$$

Thus, by using equation 36 and the Liénard-Wiechert expression for the vector potential, we obtain:

$$\vec{E} = \frac{\mu_0 p_0\omega^2}{4\pi r}\left\{\cos\Big[\omega(t-r/c)\Big]\Big(\hat{x}-\frac{x}{r}\hat{r}\Big)+\sin\Big[\omega(t-r/c)\Big]\Big(\hat{y}-\frac{y}{r}\hat{r}\Big)\right\} \tag{39}$$

$$\vec{B} = \frac{1}{c}\hat{r} \times \vec{E} \tag{40}$$

Now, we attempt to calculate the Poynting vector to study the radiation and power of this dipole. Clearly, we use the field components that vary as $\frac{1}{r}$ since their product is of order $\frac{1}{r^2}$ and the resulting integral of the Poynting vector is finite and nonzero. Considering that the fields are transverse to the direction of propagation, we have:

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \frac{E^2}{\mu_0 c}\hat{r} \tag{41}$$

By substituting the values and keeping the aforementioned orders, we obtain:

$$\vec{S} = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2 \left\{1 - \left(\sin\theta\, \cos\big[\omega(t - r/c) - \phi\big]\right)^2\right\}\hat{r} \tag{42}$$

$$\langle\vec{S}\rangle = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2 \left(1 - \frac{1}{2}\sin^2\theta\right)\hat{r} \tag{43}$$

Thus, we have:

$$P = \frac{\mu_0 p_0^2\omega^4}{16\pi^2 c}\, 2\pi \left[\int_0^{\pi} \sin\theta\, d\theta - \frac{1}{2}\int_0^{\pi} \sin^3\theta\, d\theta\right] = \frac{\mu_0 p_0^2\omega^4}{8\pi c}\left(2 - \frac{1}{2}\cdot\frac{4}{3}\right) = \frac{\mu_0 p_0^2\omega^4}{6\pi c} \tag{44}$$

4 Examination of the Equipotential Surfaces of an Ideal Rotating Dipole

To examine the equipotential surfaces of the rotating dipole, we use the general equation 34, that is, we no longer consider the third approximation and the potentials derived from it, but rather examine the more general case in the equations. Assume that we have an equipotential surface whose potential is a constant V_0. By using the sine addition formula and the identity

$$a\sin(x) + b\cos(x) = \sqrt{a^2+b^2}\,\sin\Big(x + \arctan\frac{b}{a}\Big),$$

the equation for the equipotential surface becomes:

$$r = \frac{p_0}{4\pi\epsilon_0 V_0}\sqrt{\frac{1}{r^2}+\frac{\omega^2}{c^2}}\,\sin\Big(\omega t - \frac{\omega r}{c} - \phi - \arctan\frac{c}{\omega r}\Big)\sin\theta \tag{45}$$

In this equation, r, θ, and ϕ are the spherical coordinate parameters.

Now, we wish to plot this curve. For plotting purposes, we consider two practical cases:

1. $r \gg \lambda$
2. $r \ll \lambda$

Furthermore, for ease of visualization, we plot the curves in two dimensions on the dipole plane, i.e. we plot the equipotential lines in the plane $z = 0$.

4.1 $r \gg \lambda$

$$x^2 + y^2 = \frac{p_0^2\omega^2}{32\pi^2\epsilon_0^2 V_0^2 c^2} - \frac{p_0^2\omega^2}{32\pi^2\epsilon_0^2 V_0^2 c^2}\cos\Big(2\omega t - \frac{2\omega}{c}\sqrt{x^2+y^2} - 2\arctan\frac{y}{x}\Big) \tag{46}$$

We plot the graph corresponding to this equation; these graphs are shown in the figures below. In each figure there are three panels, with each panel showing the equipotential lines at a specific time. In fact, each figure represents a specific angular velocity, and the graphs in each panel correspond to a specific time associated with that angular velocity.

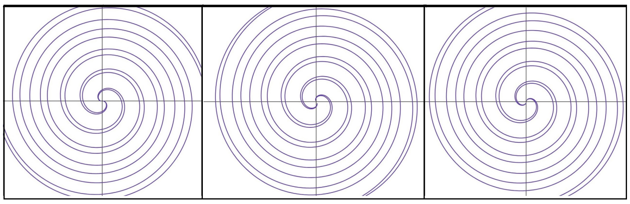

Figure 1: Equipotential curves for a given potential with angular velocity case 1 at three different times

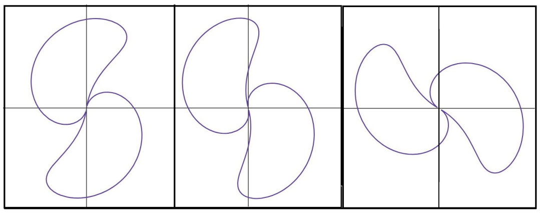

Figure 2: Equipotential curves for a given potential with angular velocity case 2 at three different times

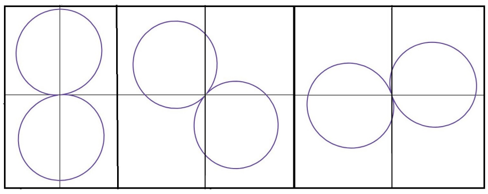

Figure 3: Equipotential curves for a given potential with angular velocity case 3 at three different times

As can be seen from the figures above, it is evident that this result is consistent with the Python code "Simulation No. 1"; that is, the equipotential lines lie on spiral curves. In the 3D simulation "Simulation No. 1", if we view from above, we see the same equipotential curves. In fact, the equipotential points—like peaks, valleys, etc.—lie on the wavefronts. Direct calculation of the propagation speed of these waves is rather cumbersome and time-consuming due to the complex radial dependence, but according to Maxwell's equations and the d'Alembertian for the potential, the electric potential in free space for the time-varying case (e.g. an oscillator) is expected to propagate at the speed of light. In free space, where the charge density is zero, the d'Alembertian of the potential vanishes, i.e.,

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0$$

which is the well-known wave equation implying that the propagation speed for the time-varying potential is equal to the speed of light.

Similarly, the Python code and the animation video corresponding to the equipotential curves, titled "Simulation No. 2", are attached for access and viewing.

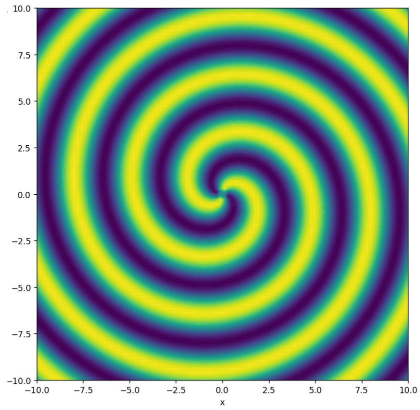

Figure 4: Equipotential curve for a given potential with a specified angular velocity at a fixed time

4.2 $r \ll \lambda$

$$x^2 + y^2 = \frac{p_0}{4\pi\epsilon_0}\cos\left(\omega t - \frac{\omega\sqrt{x^2+y^2}}{c}\right) \tag{47}$$

If we take the approximation to first order in the small parameter, we have:

$$x^2 + y^2 = \frac{p_0}{4\pi\epsilon_0}\Big(\cos(\omega t) + \frac{\omega\sqrt{x^2+y^2}}{c}\sin(\omega t)\Big) \tag{48}$$

Using the identity

$$a\sin(x) + b\cos(x) = \sqrt{a^2+b^2}\,\sin\Big(x + \arctan\frac{b}{a}\Big)$$

and the above approximation, we obtain:

$$x^2 + y^2 = \frac{p_0}{4\pi\epsilon_0}\cos(\omega t) \tag{49}$$

which is the equation of a circle with a variable radius. This is similar to a time-varying monopole. This result is logical because, according to the approximation (i.e. if the measurement distance is very small compared to the emitted wavelength, we are very close to the dipole), and given its ideal nature, one expects the equipotential surface to be circular and symmetric.

Another important case is when the angular velocity is zero, i.e. when the dipole is stationary. The graph for this case, according to equation 45, is as follows:

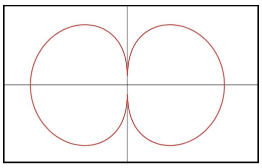

Figure 5: Equipotential curve of a stationary dipole for a given potential

As we can see, these figures are astonishingly related; for example, note that the model in figure 5 is simply a rotated and stretched version of the curves in figure 2!

5 Conclusion and Other Ideas

The comparison of the equipotential curves for an ideal stationary dipole and an ideal rotating dipole demonstrates why, in the past, scientists were fascinated by the ether theory. In fact, if we enter the rotating frame attached to the body, there appears to be a rotating ether wind (an ether vortex) that seems to drag the equipotential lines along with it! Intuitively, this phenomenon is very intriguing.

Of course, there are other classical perspectives on the subject. For example, one may consider the equipotential surfaces as a single non-rigid (gel-like) body undergoing rotational motion—a different way of examining this idea. Due to the inertia of such rotating gel-like bodies, they deform. One approach to studying the material properties in these two ideas is to write down the relevant equations. For instance, in the ether case, by writing these equations and using fluid dynamics, one might expect to calculate the material properties of the ether to some extent. Similarly, for the gel-like body case, by assuming the validity of classical mechanics and considering a frame attached to the rotating dipole while analyzing the allowed forces in the system, calculating the stress tensor, and using various techniques in fluid and solid mechanics along with a proper definition of Young's modulus, one could deduce the hypothetical properties of these materials necessary for establishing a correct electromagnetic theory.

In other words, just as physicists once attempted to reconcile electromagnetic theory with experiments by introducing the ether and then adjusting its properties, here one might also follow similar procedures. However, due to space limitations in this article and the voluminous calculations involved, we merely mention these ideas without carrying out the detailed computations. In fact, the plotted graphs make the introduction of something like the ether, as it was conceived in the 18th and 19th centuries, quite understandable—and they explain why there was such insistence on this theory, as it was in some way consistent with everyday experience.

From a relativistic viewpoint, many further investigations can be carried out in this article. For example, we know the four-dimensional transformations of the potentials, as well as the field transformations via the antisymmetric field tensor; thus, we can write these transformations and, by doing so, obtain the transformation of the shapes of these fields. This is very interesting because the transformation of physical shapes in relativity can be easily obtained in two different reference frames via Lorentz transformations. Here, the transformation of the shape of the two potential fields (i.e. the equipotential surfaces) can be obtained using relativistic transformations! Again, due to space limitations, we only mention this possibility without detailed calculations.

In general, it appears that electromagnetic theory opens vast avenues for research and inquiry as well as new branches in physics, and by mastering it correctly, one can gradually step into the grand world of physics and its research.